Math 565: Functional Analysis
Lecture 20

Prop (norm = max inner prod.). Let H be an inner prod. space. For each $x \in H$, $\|x\| = \max_{\|y\| \leq 1} |\langle x, y\rangle| = \max_{\|y\|=1} |\langle x, y\rangle|$.

Proof. For any $\|y\| \leq 1$, CBS gives $|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \leq \|x\|$, so the max of $|\langle x, y\rangle|$ is achieved by $y := \frac{1}{\|x\|} x$ (assuming $x \neq 0$): indeed, $|\langle x, \frac{1}{\|x\|} x\rangle| = \frac{1}{\|x\|} \langle x, x\rangle = \|x\|$. $\square$

Recall that in any normed vector space, the function $x \mapsto \|x\|$ is continuous, in fact, 1-Lipschitz: $|\|x\| - \|y\|| \leq \|x - y\|$. In an inner product space, we also have (via Cauchy-B-Schwartz):

Prop. In an inner prod. space H, the map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ is continuous.

Proof. Let $x_n \to x$, $y_n \to y$ and note that

$$|\langle x, y\rangle - \langle x_n, y_n\rangle| \leq |\langle x, y\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x_n, y_n\rangle| = |\langle x - x_n, y\rangle| + |\langle x_n, y - y_n\rangle| \overset{\text{CBS}}{\leq} \|x - x_n\| \cdot \|y\| + \|x_n\| \cdot \|y - y_n\|$$

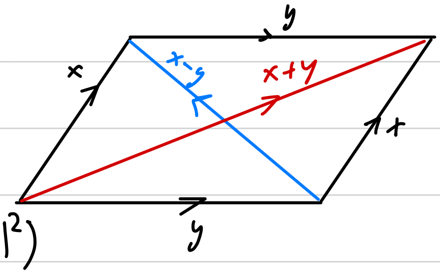
$\to 0$ as $n \to \infty$ because $\|x_n\| \to \|x\|$ and $\|x - x_n\|, \|y_n - y\| \to 0$. $\square$

Prop. Let H be an inner prod. space. Then the following hold:

(a) Parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

(b) Polarization identity: $\langle x, y\rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|x + i^k y\|^2$

$$= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) + \frac{1}{4} i \left(\|x+iy\|^2 - \|x-iy\|^2\right)$$



Theorem. An normed vector space is an inner product space $\Leftrightarrow$ parallelogram law holds. In this case, the inner product is defined by the polarization identity.

Proof. Optional HW.

Orthogonal projections.

Theorem. Let K be a closed convex subset of a Hilbert space H. Then for each $h \in H$ there is a unique $k_0 \in K$ of minimum distance from h, i.e. $\|h - k_0\| = \inf_{k \in K} \|h - k\|$.

Proof. By shift K and h by $-h$, we may assume WLOG that $h=0$. Put $d := \inf_{k\in K} \|k\|$.
Note that by the parallelogram law and convexity, for any $k, k' \in K$, we have:

$$\frac{1}{4}\|k-k'\|^2 = \|\tfrac{1}{2}k - \tfrac{1}{2}k'\|^2 = \frac{1}{2}\|k\|^2 + \frac{1}{2}\|k'\|^2 - \|\tfrac{1}{2}k + \tfrac{1}{2}k'\|^2 \le \frac{1}{2}\|k\|^2 + \frac{1}{2}\|k'\|^2 - d^2 \quad (*)$$

Existence. Let $(k_n) \subseteq K$ be s.t. $\|k_n\| \to d$ as $n \to \infty$. Then by $(*)$,

$$\frac{1}{4}\|k_n - k_m\|^2 \le \frac{1}{2}\|k_n\|^2 + \frac{1}{2}\|k_m\|^2 - d^2 \to 0 \quad \text{as } \min(n,m) \to \infty,$$

so (k_n) is Cauchy, hence $k_n \to$ some $k_0 \in K$ by the completeness of H and closedness of K.
By the continuity of $x \mapsto \|x\|$, $\|k_0\| = \lim_{n\to\infty} \|k_n\| = d$.

Uniqueness. Let $k_0' \in K$ be s.t. $\|k_0'\| = d$. Then again by $(*)$,

$$\frac{1}{4}\|k_0 - k_0'\| \le \frac{1}{2}\|k_0\| + \frac{1}{2}\|k_0'\| - d^2 = 0,$$

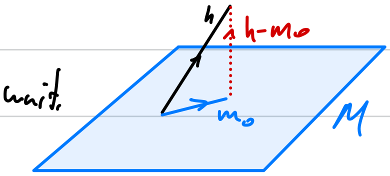
so $k_0 = k_0'$. □

For a subset A of an inner prod. sp. H and $h \in H$, write $h \perp A$:<=> $h \perp a$ for all $a \in A$.
Denote $A^\perp := \{h \in H : h \perp A\}$.

Prop. $A^\perp$ is a closed subspace of H and $A^\perp = (\overline{\operatorname{span} A})^\perp$

Proof. That $A^\perp$ is a subspace and $A^\perp = (\operatorname{span} A)^\perp$ follow from the bilinearity of the inner product. Furthermore $A^\perp = \bigcap_{a\in A} \{h \in H : \langle h, a\rangle = 0\}$ so $A^\perp$ is closed by the continuity of $\langle \cdot, a\rangle$.
Similarly, $(\operatorname{span} A)^\perp = (\overline{\operatorname{span} A})^\perp$ by the continuity of $\langle h, \cdot\rangle$. □

Lemma. Let H be an inner prod. space and $M \subseteq H$ a subspace. Then for each $h \in H$, $m_0 \in M$,
m_0 minimizes the $\operatorname{dist}(h, M) \iff h - m_0 \perp M$.



Proof. $\Rightarrow$. Let $m \in M$ be a nonzero vector, which we may assume to be a unit.
Let $x := h - m_0$. Want: $\langle x, m\rangle = 0$, equiv., $\operatorname{proj}_m x := \langle x, m\rangle m = 0$.
But $x = \operatorname{proj}_m x + (x - \operatorname{proj}_m x)$ so by Pythagoras, $\|\operatorname{proj}_m x\|^2 = \|x\|^2 - \|x - \operatorname{proj}_m x\|^2$ and $\|x - \operatorname{proj}_m x\| = \|h - (m_0 + \langle x, m\rangle m)\| \ge \|h - m_0\| = \|x\|$, so $\|\operatorname{proj}_m x\|^2 = \|x\|^2 - \|x\|^2 = 0$.

$\Leftarrow$. Suppose $h - m_0 \perp M$. Then $\forall\, m \in M$, $\|h - m\|^2 = \|h - m_0 + (m_0 - m)\|^2 = \|h - m_0\|^2 + \|m_0 - m\|^2 \ge \|h - m_0\|^2$,
so m_0 is indeed closest to h in M, i.e. $\|h - m_0\| = \inf_{m\in M} \|h - m\|$. □

Cor. Let H be a Hilbert space and $M \subseteq H$ be a closed subspace. Then for each $h \in H$ there is a unique vector in M, denoted $\mathrm{proj}_M h$ and called the (orthogonal) projection of h onto M, such that $h - \mathrm{proj}_M h \perp M$; equiv. $\mathrm{proj}_M h$ achieves $\mathrm{dist}(h, M)$.

Cor. Let H be a Hilbert space and $M \subseteq H$ a closed subspace. Then every $h \in H$ uniquely decomposes as $h = \mathrm{proj}_M h + \mathrm{proj}_{M^\perp} h$. In particular, $H = M + M^\perp$.

Proof. We only need to show that $x := h - \mathrm{proj}_M h = \mathrm{proj}_{M^\perp} h$. By the uniqueness of the projection, we only need to check that $h - x \perp M^\perp$. But $h - x = \mathrm{proj}_M h \in M$ so $h - x \perp M^\perp$ by def. □

Cor. Let H be a Hilbert space and $M \subseteq H$ a closed subspace. Then $(M^\perp)^\perp = M$.
In particular, $(A^\perp)^\perp = \overline{\mathrm{span}\, A}$ for any subset $A \subseteq H$.

Proof. For each $h \in H$, we have $\mathrm{proj}_M h + \mathrm{proj}_{M^\perp} h = h = \mathrm{proj}_{M^\perp} h + \mathrm{proj}_{(M^\perp)^\perp} h$, so $\mathrm{proj}_M = \mathrm{proj}_{(M^\perp)^\perp}$, hence $M = (M^\perp)^\perp$. □

Riesz representation.

For $y \in H$, note that $f_y : H \to \mathbb{C}$ by $x \mapsto \langle x, y \rangle$ is a bdd linear functional. Indeed, $|f_y(x)| = |\langle x, y\rangle| \leq \|y\| \|x\|$, so $\|f_y\| \leq \|y\|$. In fact, $\|f_y\| = \|y\|$ because we can take $x := \frac{1}{\|y\|} \cdot y$ (when $y \neq 0$), so $|f_y(x)| = |\langle x, y \rangle| = \frac{1}{\|y\|} \langle y, y \rangle = \|y\|$. In other words, $y \mapsto f_y : H \to H^*$ is a conjugate-linear isometry because $\alpha y \mapsto \bar{\alpha} f_y$.

Riesz rep. thm. For any Hilbert space H, the map $\iota : H \to H^* : y \mapsto f_y$ is an isometric conjugate-linear isomorphism.

Proof. We just need to prove surjectivity. Let $0 \neq f \in H^*$. Then $M := \ker f$ is a closed proper subspace (this is the only place where the continuity of f is used). Since $H = M + M^\perp$, $M^\perp \neq \{0\}$, so $\exists$ unit $u \in M^\perp$. For each $x \in H$, $f(x - \frac{f(x)}{f(u)} u) = 0$, so $x - \frac{f(x)}{f(u)} u \in M$, hence $\langle x - \frac{f(x)}{f(u)} u, u \rangle = 0$, i.e. $0 = \langle x - \frac{f(x)}{f(u)} u, u \rangle = (x, u) - \frac{f(x)}{f(u)} \langle u, u \rangle = \langle x, u \rangle - \frac{f(x)}{f(u)}$. Thus, $f(x) = f(u) \langle x, u \rangle = \langle x, \overline{f(u)} u \rangle$, so $f = f_y$ where $y := \overline{f(u)} u$. □